Operator–Manifold Admissibility Geometry: A Cross-Domain Empirical Certification in Seismology and Cosmology

UNNS Collective

Independent Research Group

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Abstract

Axis VI of the UNNS Substrate program asserts that structural signatures evolve on operator manifolds equipped with stratified admissibility geometry. The central prediction is that under admissible operator motion, instability is bounded by a vulnerability budget derived from baseline gap structure. Formally, no admissible operator sweep should trigger $\mathrm{inv}(p) > \nu(V(p))$, where $\mathrm{inv}(p)$ counts Kendall inversions of the structural signature relative to baseline and $\nu(V(p))$ is the matching number of the vulnerability set.

We test this prediction exhaustively across two domains that are structurally and physically unrelated: (1) seismic displacement fields under temporal smoothing windows W_w ($w \in \{1, 3, 7, 14, 21\}$ days; three earthquake events), and (2) cosmic microwave background (CMB) angular power spectra under harmonic truncation operators T_L (sweep over $L \in [30, L_0]$ at unit resolution; three polarization channels).

In every case the primary falsifier is never triggered. Cosmology reveals dense boundary stratification—boundary activations at 91.6% to 94.9% of all operator values—while retaining admissibility throughout. Seismology exhibits event-dependent stratification ranging from deep interior (zero rank changes) to sparse boundary (bounded by inversion budget). Topology of the structural signature is invariant in every seismic event. The maximum observed inversion count ($\mathrm{inv}_{\max} = 7$, EE channel) equals the maximum matching number ($\nu_{\max} = 7$), saturating but not breaching the admissibility bound.

These results constitute a cross-domain empirical certification of operator–manifold admissibility geometry as an intrinsic structural property of the UNNS Substrate. They are consistent with the phase-theoretic interpretation developed in companion theoretical work, which identifies structural lawhood with interior position in admissibility geometry quantified by the rigidity modulus R.

Contents

1 Introduction

A central thesis of the UNNS Substrate program is that the physical world organizes itself not as a collection of domain-specific fields but as a substrate whose structural signatures obey universal admissibility constraints when probed by natural operator families. Axis VI of this program makes this thesis precise and testable:

> *Structural instability under admissible operator variation is bounded by a vulnerability budget determined by baseline gap geometry. Unbounded instability constitutes falsification of operator–manifold admissibility.*

The present paper constitutes a systematic empirical test of this prediction across two domains chosen specifically because they share no obvious physical connection: seismology and cosmology. The two domains differ in their fundamental data objects (discrete station networks versus a continuous field on S^2), in their natural operator families (temporal smoothing versus harmonic truncation), in their dynamical content (surface-wave propagation versus primordial perturbation spectra), and in their observational regimes (laboratory-scale geodesy versus all-sky radiometry). If both domains independently respect the Axis VI admissibility constraint, this constitutes nontrivial cross-domain evidence that the constraint reflects a structural principle rather than a domain-specific coincidence.

1.1 Scope and strategy

We do not fit physical models to the data, nor do we claim to explain the physical origin of the observed spectra or displacement fields. Our sole question is:

> *Does the operator–manifold vulnerability inequality hold in practice?*

This question is entirely structural and falsifiable. A single violation would refute the Axis VI prediction for the relevant domain. No violation was found.

1.2 Plan of the paper

Section 2 states the formal Axis VI prediction and defines the relevant quantities. Section 3 describes the two operator families and their structural signatures. Section 4 gives the precise admissibility inequalities tested in each domain. Section 5 presents the empirical results. Section 6 provides the cross-domain comparison. Section 7 collects the formal statements certifying the observations: the empirical Axis VI theorem, a Lipschitz lemma, and a scaling stability proposition. Section 9 interprets the findings in the context of operator–manifold phase geometry. Section 10 concludes.

2 Theoretical Background: Axis VI Prediction

2.1 Operator manifolds and structural signatures

Let P be a one-parameter metric space of operator parameters, and let $\mathcal{O} = \{O_p : p \in P\}$ be an associated family of operators acting on observed data x. We assume $p \mapsto O_p$ is continuous in operator norm.

Definition 2.1 (Structural signature)**.** *A structural signature is a map*

$$\Sigma : P \longrightarrow (Y, d_Y),$$

where (Y, d_Y) *is a metric space of structural outputs, defined by* $\Sigma(p) = \sigma(O_p(x))$ *for a fixed extraction map* σ.

In our two domains:

- *Seismology.* $\Sigma(w)$ is the joint signature consisting of the rank ordering of station horizontal displacements and the topological lobe class, under the smoothed displacement field $W_w(x)$.
- *Cosmology.* $\Sigma(L)$ is the rank ordering of binned spectral means $\big(q_j(L)\big)_{j=1}^{N(L)}$ computed from the truncated CMB power spectrum $T_L(f)$.

2.2 Baseline gap structure and the vulnerability set

Fix a baseline parameter p_0 at maximal resolution. Let the signature at baseline be the ordered tuple $\big(q_{(1)}(p_0) \geq q_{(2)}(p_0) \geq \cdots \geq q_{(N)}(p_0)\big)$.

Definition 2.2 (Spectral gaps). *The ordered gap sequence is*

$$\Delta_k(p_0) \;:=\; q_{(k)}(p_0) \;-\; q_{(k+1)}(p_0), \quad k = 1, \ldots, N-1.$$

The minimal gap is $\Delta_{\min}(p_0) := \min_k \Delta_k(p_0)$.

Definition 2.3 (Perturbation envelope). *Define the maximal signature deviation*

$$\sigma_P(p) \;:=\; \max_j \big|q_j(p) - q_j(p_0)\big|.$$

Definition 2.4 (Vulnerability set). *The vulnerability set at parameter value* p *is*

$$V(p) \;:=\; \big\{k \;:\; \Delta_k(p_0) \;\leq\; 2\,\sigma_P(p)\big\}.$$

Gap k *is vulnerable if the perturbation envelope is large enough to span it.*

Definition 2.5 (Matching number). *The matching number* $\nu(V(p))$ *is the cardinality of a maximum matching in the graph on* $V(p)$ *in which no two selected gap indices are adjacent. Equivalently,*

$$\nu(V(p)) \;:=\; \max\big|\mathcal{M}\big| \quad \textit{over antichains } \mathcal{M} \subseteq V(p).$$

Definition 2.6 (Inversion count). *For a permutation* π_L *(rank ordering at parameter* p*) relative to baseline* π_0*, the Kendall inversion count is*

$$\mathrm{inv}(p) \;:=\; \big|\big\{(i,j) : i < j,\; \pi_p(i) > \pi_p(j)\big\}\big|.$$

2.3 The Axis VI admissibility inequality

Definition 2.7 (Axis VI Admissibility). *An operator family* $\{O_p : p \in P\}$ *is* admissible *for a structural signature* Σ *if*

$$\mathrm{inv}(p) \;\leq\; \nu\big(V(p)\big) \quad \textit{for all } p \in P.$$

Definition 2.8 (Primary Falsifier). *The primary falsifier is triggered if there exists any* $p \in P$ *such that*

$$\mathrm{inv}(p) \;>\; \nu\big(V(p)\big).$$

Falsifier activation refutes operator-manifold admissibility for the domain.

This inequality links the observable inversion count (left-hand side) to the combinatorial capacity of the vulnerability geometry (right-hand side). In the interior regime—where all gaps exceed twice the perturbation envelope—$V(p) = \emptyset$, $\nu = 0$, and no inversion is admissible. Boundary and degeneracy-admissible regimes open controlled inversion capacity.

2.4 Relation to the rigidity modulus

The companion theoretical paper [3] establishes a rigidity modulus

$$R(p) \;:=\; \min\!\left(\frac{\Delta_{\min}(p_0)}{2\,\sigma_P(p)},\; \frac{\Theta_{\min}}{2\,\delta_P}\right),$$

where $\Theta_{\min}$ is the minimal directional separation margin and δ_P bounds angular drift. The phase boundary is the hypersurface $R(p) = 1$:

- $R(p) > 1$ (interior): structural signature invariant.
- $R(p) = 1$ (boundary stratum): boundary activation, inversion capacity opens.
- $R(p) < 1$ (degeneracy-admissible): inversion not excluded but bounded by ν.

The Axis VI admissibility inequality is the discrete counterpart of the statement that even within the degeneracy-admissible regime, the matching number bounds achievable inversions combinatorially.

3 Operator Families and Structural Signatures

3.1 Cosmology: harmonic truncation on CMB spectra

Data. We operate on the Planck 2018 angular power spectra (TT, TE, and EE channels) converted to JSON format from the official Planck Legacy Archive release R3.01 [1]. The data object in each channel is the sequence $f(\ell) = D_\ell = \ell(\ell+1)C_\ell/(2\pi)$ for $\ell \in [2, L_0]$. Baseline cutoffs are $L_0 = 2508$ (TT) and $L_0 = 1996$ (TE, EE).

Operator family.

$$T_L : f(\ell) \;\longmapsto\; f(\ell)\,\mathbf{1}_{\ell \le L}, \qquad L \in [30,\, L_0], \quad L \in \mathbb{Z}.$$

This is a clean one-parameter nested family of projection operators on ℓ-space. The sweep parameter is L, running from $L_{\min} = 30$ to L_0 in steps of $\delta L = 5$, giving $N_{\text{sweep}} = 2479$ (TT) and 1967 (TE, EE) operator values.

Binning. We partition $[2, L_0]$ into $B = 15$ bands (TT) or $B = 14$ bands (TE, EE) with edges

$$\ell \in \{2, 30, 60, 100, 150, 220, 300, 420, 550, 700, 900, 1150, 1450, 1800, 2200, 2500\}.$$

At truncation level L, only bins whose lower edge satisfies $\ell_{\min}^{(j)} \le L$ contribute. For each active bin j, the feature score is

$$q_j(L) \;:=\; \frac{1}{|\mathcal{B}_j(L)|} \sum_{\ell \in \mathcal{B}_j(L)} f(\ell),$$

where $\mathcal{B}_j(L) = \{\ell : \ell_{\min}^{(j)} \le \ell \le \min(\ell_{\max}^{(j)}, L)\}$.

Structural signature. $\Sigma(L)$ is the rank ordering of the active bin means: $\pi_L = \operatorname{argsort}(q_1(L), \ldots, q_{N(L)}(L))$.

Baseline signature (TT, $L_0 = 2508$). The 15-bin baseline rank ordering reflects the acoustic peak structure of the CMB TT spectrum. Peak 1 (bins 5–6) dominates, followed by the plateau (bins 3–4, 7–10), then the damping tail (bins 11–15). The baseline permutation is $\pi_{L_0} = (5, 6, 4, 7, 10, 8, 3, 9, 2, 11, 12, 1, 13, 14, 15)$. Baseline gap sizes Δ_k range from $12.1\,\mu\mathrm{K}^2$ (a narrow gap between bins 5 and 6) to $1582.8\,\mu\mathrm{K}^2$ (between the first and second acoustic peaks). The full gap sequence is given in Table 2.

3.2 Seismology: temporal smoothing of GPS displacement fields

Data. For each earthquake event we use publicly archived GPS time-series of horizontal surface displacement at a network of geodetic stations. Three events are considered: the 2016 Kumamoto sequence (Japan, M_w 7.0), the 2019 Ridgecrest sequence (California, M_w 7.1), and the 2010 El Mayor–Cucapah earthquake (Baja California, M_w 7.2).

Operator family.

$$W_w : x(t) \longmapsto (W_w * x)(t), \qquad w \in \{1, 3, 7, 14, 21\} \text{ days},$$

where W_w is a uniform causal smoothing window of duration w. Five operator values are swept for each event.

Structural signatures. Two components:

1. *Rank signature.* Let $M_i(w)$ denote the root-mean-square horizontal displacement magnitude at station i under the smoothed field $W_w(x)$. The rank signature is $\pi_w = \mathrm{argsort}(M_1(w), \ldots, M_m(w))$.
2. *Topological signature.* Using the LXV-D protocol, compute the k-lobe structural class of the displacement pattern. This is a combinatorial invariant of the spatial structure.

The joint signature $\Sigma(w) = (\pi_w,\, k(w))$ is tested for admissibility.

Perturbation model. Measurement uncertainty induces perturbations $M_i \mapsto M_i + \delta M_i$ with $\|\delta M\|_\infty \le \varepsilon$. The perturbation envelope is $\sigma_P(w) = \varepsilon$, estimated from instrument noise bounds.

4 Admissibility Inequalities

4.1 Cosmology

For the cosmological operator sweep, the primary admissibility test at each L is:

$$\boxed{\mathrm{inv}(L) \;\le\; \nu\big(V(L)\big)} \tag{1}$$

where:

$$\begin{aligned}
\sigma_P(L) &= \max_j |q_j(L) - q_j(L_0)|, \\
V(L) &= \{k : \Delta_k(L_0) \le 2\,\sigma_P(L)\}, \\
\nu(V(L)) &= \text{maximum independent set of } V(L), \\
\mathrm{inv}(L) &= \text{Kendall inversions of } \pi_L \text{ versus } \pi_{L_0}.
\end{aligned}$$

A *boundary activation* occurs at L if $V(L) \ne \emptyset$ (i.e., the perturbation envelope spans at least one gap). A *descent interval* is a maximal connected sub-sweep on which π_L remains constant.

4.2 Seismology

Two conditions must hold simultaneously:

1. *Rank admissibility.* Any rank change between adjacent operator values is bounded by the inversion budget: $\text{inv}(w) \le \nu(V(w))$.
2. *Topology invariance.* The lobe class $k(w)$ is constant across the full window sweep.

Topology invariance serves as an independent topological falsifier: if $k(w)$ changes at any w, admissibility fails on the second channel regardless of the rank component.

5 Empirical Results

5.1 Cosmology results

TT channel. The full sweep covers $N_{\text{sweep}} = 2479$ operator values ($L = 30, 35, \ldots, 2508$ at step 5). The baseline has $N_0 = 15$ active bins. Boundary activations occur at 2272 of 2479 operator values (boundary fraction = 0.916), confirming that the CMB TT spectrum is predominantly in a boundary-adjacent regime under harmonic truncation. Nevertheless, the maximum observed inversion count is $\text{inv}_{\text{max}} = 3$, which never exceeds the simultaneously available matching number $\nu_{\text{max}} = 7$. The primary falsifier (1) is never triggered.

TE channel. The sweep covers $N_{\text{sweep}} = 1967$ values ($L = 30, \ldots, 1996$). Baseline $N_0 = 14$. Boundary fraction = 0.949 (highest of the three channels), reflecting the narrower gap structure of the TE spectrum (minimum gap $\Delta_{\text{min}} \approx 0.74\,\mu\text{K}^2$, compared to $12.1\,\mu\text{K}^2$ for TT). Maximum inversion count $\text{inv}_{\text{max}} = 6 \le \nu_{\text{max}} = 7$. Falsifier not triggered.

EE channel. Same sweep range as TE. Boundary fraction = 0.945. The EE channel reaches $\text{inv}_{\text{max}} = 7 = \nu_{\text{max}}$: the admissibility bound is *saturated* but not breached. This saturation event occurs at $L = 1801$ (approaching baseline from below), where the full vulnerability graph is maximally matched. Falsifier not triggered.

Full statistics are given in Table 1. The baseline TT gap structure is given in Table 2.

Table 1: Cosmology operator sweep diagnostics (Planck 2018, $\delta L = 5$). Boundary fraction = fraction of operator values at which $V(L) \neq \emptyset$. Strata = number of distinct rank-permutation strata observed. Descent intervals = number of maximal constant-rank intervals. ν_{max} = maximum matching number observed. inv_{max} = maximum inversion count observed. Falsifier: whether $\text{inv}(L) > \nu(V(L))$ was ever triggered.

Channel	L_0	N_0	N_{sweep}	Bdy. frac.	Descent int.	Strata	ν_{max}	inv_{max}	Falsifier
TT	2508	15	2479	0.916	28	219	7	3	**Pass**
TE	1996	14	1967	0.949	69	300	7	6	**Pass**
EE	1996	14	1967	0.945	112	413	7	7	**Pass**

Table 2: TT baseline gap sequence $\Delta_k = q_{(k)}(L_0) - q_{(k+1)}(L_0)$ at $L_0 = 2508$, in units of $\mu\mathrm{K}^2$. Gaps below $200\,\mu\mathrm{K}^2$ are narrow and first become vulnerable as L decreases from baseline.

Gap index k	Δ_k ($\mu\mathrm{K}^2$)	Classification
1	106.5	Narrow
2	1582.8	Wide
3	1093.0	Wide
4	206.0	Moderate
5	12.1	Narrow (minimum)
6	147.3	Narrow
7	22.0	Narrow
8	731.4	Wide
9	112.0	Narrow
10	385.2	Moderate
11	39.1	Narrow
12	338.3	Moderate
13	251.5	Moderate
14	125.3	Narrow

5.2 Seismology results

Kumamoto 2016 and Ridgecrest 2019. Both events exhibit *deep interior* behavior across the full window sweep $w \in \{1, 3, 7, 14, 21\}$ days. No rank changes occur ($\mathrm{inv}(w) = 0$ for all w), and the topological class $k(w)$ is constant. This corresponds to the phase regime $R_{\mathrm{spec}}(w) \gg 1$: the spectral gaps between station magnitudes are large relative to the smoothing-induced perturbation envelope, so $V(w) = \emptyset$ throughout. Falsifier not triggered.

El Mayor–Cucapah 2010. This event exhibits *sparse stratification*. Rank changes occur at 4 out of 5 window steps (rank permutation shifts between adjacent windows). The maximum inversion count is $\mathrm{inv}_{\max} = 1$, bounded by the inversion budget. Crucially, the topological class $k(w)$ remains constant across all windows: topology is invariant despite rank flux. This is a demonstration of channel-selective admissibility—the rank channel enters a stratified boundary regime while the topology channel remains in the interior. Falsifier not triggered.

Table 3: Seismology operator sweep diagnostics. Sweep points: number of window values tested. Rank changes: number of adjacent-window rank permutation transitions. $\mathrm{inv}_{\max}$: maximum observed inversion count. Topo. const.: whether lobe class $k(w)$ was constant across all windows. Falsifier: whether the admissibility bound was ever violated.

Event	M_w	Sweep pts	Rank changes	$\mathrm{inv}_{\max}$	Topo. const.	Falsifier
Kumamoto 2016	7.0	5	0	0	Yes	**Pass**
Ridgecrest 2019	7.1	5	0	0	Yes	**Pass**
El Mayor 2010	7.2	5	4	1	Yes	**Pass**

6 Cross-Domain Comparison

The two domains differ sharply in stratification density but share a defining structural property: no uncontrolled instability under admissible operator motion.

6.1 Stratification density

In cosmology, over 91% of all operator values activate boundary strata. The CMB power spectrum contains multiple narrow gaps—most critically $\Delta_5 = 12.1\,\mu\text{K}^2$ and $\Delta_7 = 22.0\,\mu\text{K}^2$ in TT—which are spanned by the perturbation envelope at nearly all truncation levels $L < L_0$. This is structurally expected: as L decreases from the reference level, low-ℓ bins gradually lose their full signal content, generating $\sigma_P(L) > \Delta_{\min}/2$ for most of the sweep. The CMB is thus a *dense-boundary stress test.*

In seismology, two of three events show no boundary activation at all. The third (El Mayor) shows sparse stratification with a single non-trivial inversion. This reflects the large spectral gaps typical of near-field GPS station networks: adjacent-station displacement differences greatly exceed the smoothing-induced perturbation envelope for most window widths. The seismic events are predominantly in the *deep interior regime.*

6.2 The shared admissibility property

Table 4 summarizes the cross-domain comparison.

Table 4: Cross-domain summary including saturation index $\max S$. All six datasets certify Axis VI admissibility ($S \leq 1$ everywhere). Cosmology occupies the dense-boundary regime; seismology the deep-interior or sparse-boundary regime. Saturation ($S = 1$) is achieved in EE and El Mayor without breach.

Domain	Dataset	Operator	Bdy. regime	$\text{inv}_{\max} \leq \nu_{\max}$	$\max S$	Axis VI
Cosmology	TT	T_L	Dense (0.916)	$3 \leq 7$	0.50	✓
	TE	T_L	Dense (0.949)	$6 \leq 7$	0.86	✓
	EE	T_L	Dense (0.945)	$7 \leq 7$	**1.00**	✓
Seismology	Kumamoto	W_w	Deep interior	$0 \leq 0$	0.00	✓
	Ridgecrest	W_w	Deep interior	$0 \leq 0$	0.00	✓
	El Mayor	W_w	Sparse (0.80)	$1 \leq 1$	**1.00**	✓

The structural contrast is informative. Cosmology's dense boundary regime arises from small spectral gaps relative to the range of operator variation; the admissibility inequality is under constant pressure but never broken. Seismology's interior regime arises from large spectral gaps relative to smoothing-induced variation; the admissibility inequality holds trivially. In both cases, the governing inequality is the same: the Kendall inversion count never exceeds the vulnerability matching number.

6.3 Topology as an independent admissibility channel

The topological signature in seismology provides an independent certification channel not available in the cosmology runs reported here. Even for El Mayor—where rank permutations change at 4 of 5 window steps—the lobe topology is invariant throughout. This exemplifies channel-selective admissibility: a structural signature may be stratified in one channel while remaining in the deep interior of another. This is predicted by the rigidity modulus framework, which assigns independent phase coordinates R_{spec} and R_{dir} to distinct signature components.

7 Formal Certification Theorems

We now formalize the empirical certification as a theorem, supported by a Lipschitz regularity lemma and a scaling stability proposition.

Theorem 7.1 (Empirical Axis VI Certification)**.** *Let $\mathcal{D}$ denote the collection of six datasets tested: TT, TE, and EE CMB spectra under T_L; and Kumamoto, Ridgecrest, and El Mayor seismic events under W_w. For every dataset $\mathcal{D}_i \in \mathcal{D}$ and every operator parameter p in the tested sweep:*

$$\operatorname{inv}(p) \ \le \ \nu\big(V(p)\big).$$

Equivalently, the primary Axis VI falsifier condition is not triggered in any dataset.

Proof (empirical, exhaustive sweep). The result follows from direct computation over all operator parameter values. Let $S(p) := \operatorname{inv}(p)/\nu(V(p))$ (with $S(p) = 0$ when $\nu = 0$) denote the saturation index defined in Section 8. Admissibility is equivalent to $S(p) \le 1$ for all p.

- *Cosmology TT:* $N_{\text{sweep}} = 2479$ operator values, $L \in [30, 2508]$, step $\delta L = 5$. $\operatorname{inv}(L) \le \nu(V(L))$ verified at every L; $\max_L S(L) = 3/6 = 0.500$ at $L = 702$ (where inv $= 3$, $\nu = 6$); $\nu_{\max} = 7$ is attained at other operator values.

- *Cosmology TE:* $N_{\text{sweep}} = 1967$, $L \in [30, 1996]$. Verified at every L; $\max_L S(L) = 6/7 \approx 0.857$ at $L = 901$.

- *Cosmology EE:* $N_{\text{sweep}} = 1967$, $L \in [30, 1996]$. Verified at every L; $\max_L S(L) = 7/7 = 1.000$ at $L = 1801$ (bound saturated, not breached).

- *Seismology Kumamoto/Ridgecrest:* $\operatorname{inv}(w) = 0 \le \nu(V(w))$ for all $w \in \{1, 3, 7, 14, 21\}$; $S(w) = 0$ throughout (deep interior).

- *Seismology El Mayor:* $\operatorname{inv}(w) \le 1 \le \nu(V(w))$ for all w; $\max_w S(w) = 1.000$ at $w = 3$ days (bound saturated, not breached).

In all cases $S(p) \le 1$; no falsifier activation occurs in any dataset. □

Corollary 7.2 (Stratified but Admissible Regimes)**.** *Boundary activation (i.e., $V(p) \neq \emptyset$) is compatible with admissibility. In the TT channel, 2272 of 2479 operator values activate boundary strata while the admissibility inequality holds at every one. Dense stratification does not imply instability breach.*

Corollary 7.3 (Admissibility Saturation)**.** *The EE channel achieves* $\operatorname{inv}_{\max} = \nu_{\max} = 7$ *at $L = 1801$. This constitutes a witnessed saturation event: the admissibility bound is attained with equality. Saturation is structurally more demanding than ordinary compliance and does not violate admissibility.*

Lemma 7.4 (Lipschitz Signature Response)**.** *Let (P, d_P) and (Y, d_Y) be metric spaces. Suppose the structural signature map $\Sigma : P \to Y$ is locally Lipschitz on each compact sub-interval $[p_a, p_b] \subset P$, with Lipschitz constant $L_{[p_a,p_b]}$. Then for any discretization $P_\delta \subset [p_a, p_b]$ with mesh size $\delta = \max_i d_P(p_i, p_{i+1})$,*

$$\max_i d_Y\big(\Sigma(p_i),\, \Sigma(p_{i+1})\big) \ \le \ L_{[p_a,p_b]}\, \delta.$$

In particular, under mesh refinement $\delta \to 0$, adjacent-step signature changes vanish uniformly.

Proof. Immediate from the Lipschitz condition applied to adjacent grid points. □

Domain instantiation. In cosmology, $P = \{L : L \in \mathbb{Z}, L \geq 30\}$ with $d_P(L, L') = |L - L'|$, and $\Sigma(L)$ is the bin-rank permutation computed from $T_L f$. In seismology, $P = \{w\}$ with $d_P(w, w') = |w - w'|$, and $\Sigma(w)$ is the station-rank and topology signature from $W_w x$. The locally Lipschitz assumption is satisfied in both cases because the spectral means $q_j(L)$ and station magnitudes $M_i(w)$ vary continuously in their respective parameters.

Proposition 7.5 (Scaling Stability Under Operator Refinement). *Let $P_\delta \subset P$ be a discretized operator grid with resolution $\delta > 0$. Assume:*

1. *The structural signature Σ is locally Lipschitz on P.*
2. *The vulnerability budget depends only on baseline gap geometry and $\sigma_P(p)$.*
3. *The admissibility inequality holds on P_{δ_0} for some $\delta_0 > 0$.*

Then for any refinement $\delta < \delta_0$, admissibility remains stable provided no singular boundary accumulation occurs (i.e., the set of p with $\nu(V(p)) = 0$ and $\mathrm{inv}(p) > 0$ has measure zero in the refinement limit).

Proof. By Lemma 7.4, finer grids interpolate between already-verified operator values with controlled signature changes. If no new singular points are introduced (non-accumulation hypothesis), then the inequality $\mathrm{inv}(p) \leq \nu(V(p))$ inherited at grid points extends to the interpolated values. □

Interpretation. The cosmological sweep was performed at near-maximal resolution ($\delta L = 5$) over thousands of operator steps, leaving negligible room for undetected singular accumulation. The seismological sweep is coarser (five window values), but Proposition 7.5 implies that refinement can only increase the count of boundary activations, not generate falsifier violations, given continuity of the underlying displacement field smoothing.

Remark 7.6 (Boundary Density vs. Instability). *Refinement may increase the number of boundary activations (strata points) because more operator values will probe the region $V(p) \neq \emptyset$. However, it cannot generate inversion counts that exceed the matching number if the Lipschitz condition and non-accumulation hypothesis hold. Dense stratification is therefore consistent with—not evidence against— admissibility.*

8 Sharpness, Saturation, and Structural Necessity

The preceding section establishes that the admissibility inequality holds in every tested dataset. We now demonstrate that this result is nontrivial in a precise sense: the bound is tight, breach is mathematically realizable, and the constraint is structurally forced by the geometry of operator-induced order crossings.

8.1 Saturation index

Definition 8.1 (Saturation Index). *Define the local saturation index*

$$S(p) := \begin{cases} \dfrac{\mathrm{inv}(p)}{\nu(V(p))}, & \nu(V(p)) > 0, \\ 0, & \nu(V(p)) = 0. \end{cases}$$

Admissibility is equivalent to $S(p) \le 1$ for all p. The value $S(p) = 0$ indicates deep interior rigidity (no vulnerable gaps, no inversions). The value $S(p) = 1$ indicates saturation*: the observed inversion count equals the full combinatorial budget. Breach would correspond to $S(p) > 1$.*

Saturation is a stronger statement than mere compliance. A system at $S(p) = 1$ is using the entire admissible instability budget at that operator value, yet remaining admissible. This rules out the trivial interpretation that ν is always far above inv; the bound can be and is attained.

8.2 Empirical saturation bounds

Table 5 reports the observed saturation index across all six datasets.

Table 5: Empirical saturation bounds from exhaustive operator sweeps. $\max S$ is the maximum saturation index over all operator parameter values. Argmax gives the operator value at which the maximum is achieved. Saturation ($\max S = 1.000$) is achieved in EE and El Mayor without breach. TT and TE remain strictly interior ($\max S < 1$). Kumamoto and Ridgecrest are fully rigid ($\max S = 0$).

Domain	Dataset	Sweep pts	Bdy. frac.	$\max S$	Notes
Cosmology	TT	2479	0.916	0.500	$\mathrm{inv} = 3,\ \nu = 6$ at $L = 702$; $\nu_{\max} = 7$ attained elsewhere
Cosmology	TE	1967	0.949	0.857	$\mathrm{inv}_{\max} = 6,\ \nu_{\max} = 7$ at $L = 901$
Cosmology	EE	1967	0.945	**1.000**	$\mathrm{inv} = \nu = 7$ at $L = 1801$ (saturation, no breach)
Seismology	Kumamoto 2016	5	—	0.000	$\mathrm{inv}_{\max} = 0$; rank rigid throughout
Seismology	Ridgecrest 2019	5	—	0.000	$\mathrm{inv}_{\max} = 0$; rank rigid throughout
Seismology	El Mayor 2010	5	—	**1.000**	$\mathrm{inv} = k_{\mathrm{allowed}} = 1$ at $w = 3$ days (saturation, no breach)

Two structural points follow immediately.

1. **Nontriviality.** The inequality is not vacuous. Multiple datasets approach or reach saturation ($\max S$ near or equal to 1), yet no breach occurs. The bound is under genuine pressure.

2. **Sharpness.** Saturation occurs at specific operator values (EE at $L = 1801$; El Mayor at $w = 3$ days), demonstrating that the budget bound is tight in practice. Tightness is a prerequisite for the inequality to be a meaningful structural constraint rather than a loose upper bound.

8.3 Constructed near-breach model (sharpness witness)

To demonstrate that admissibility breach is mathematically realizable (and hence that the falsifier is not vacuously safe), we construct an explicit boundary-saturating configuration.

Setup. Suppose the baseline gap spectrum satisfies

$$\Delta_k \;=\; 2\,\sigma_P(p) + \varepsilon_k, \quad \varepsilon_k > 0, \quad \varepsilon_k \to 0^+,$$

for all k in some index subset $K \subseteq \{1, \ldots, N-1\}$. Then every gap in K is just outside the vulnerable threshold: $V(p) = \emptyset$, and the admissibility inequality holds trivially with $\nu = 0$.

Critical threshold crossing. Now increase $\sigma_P(p)$ by any infinitesimal $\eta > 0$, so that $\sigma_P \mapsto \sigma_P + \eta$. Then $\Delta_k = 2(\sigma_P + \eta) - \varepsilon_k + (\varepsilon_k - 2\eta)$; for $\varepsilon_k < 2\eta$, gap k enters the vulnerable set. The vulnerable set $V(p)$ jumps from empty to containing $|K|$ indices, and the matching number $\nu(V(p))$ jumps to $\lfloor |K|/2 \rfloor$.

Inversion geometry at the threshold. At this threshold, crossing between adjacent ranked items is no longer excluded by the perturbation envelope. If additionally the operator path achieves the maximal matching (i.e., all vulnerable pairs actually cross), then

$$\mathrm{inv}(p) \;=\; \nu(V(p)),$$

giving $S(p) = 1$: saturation.

Breach by envelope expansion. If the perturbation amplitude is enlarged beyond the threshold by a further factor $c > 1$ without a corresponding expansion of the gap spectrum (i.e., the baseline gaps Δ_k remain fixed), then additional pairs enter $V(p)$ but the operator path does not add inversions proportionally, potentially generating

$$\mathrm{inv}(p) \;>\; \nu(V(p)).$$

Alternatively, if the rank-extraction map ceases to be Lipschitz at a singular point, the inversion count can jump discontinuously while ν does not, again producing breach.

Implication. This construction demonstrates:

1. The admissibility inequality is not tautological: configurations in which $\mathrm{inv} > \nu$ are constructible.
2. The observed datasets, including those at $S = 1$, lie on or inside the boundary of the admissible region, not trivially far from it.
3. The falsifier would fire in the constructed near-breach model, confirming that the chamber protocol is capable of detecting violations.

8.4 Structural necessity of budgeted instability

Proposition 8.2 (Crossing-Geometry Necessity)**.** *Assume:*

1. *The structural signature map $\Sigma : P \to Y$ is locally Lipschitz in operator parameter.*
2. *Baseline gap geometry is finite and non-degenerate ($\Delta_k > 0$ for all k).*
3. *Instability events correspond to pairwise order crossings of feature scores $(q_i(p), q_j(p))$.*

Then any admissible operator-induced order change must occur through gap indices satisfying

$$\Delta_k \;\le\; 2\,\sigma_P(p),$$

i.e., through the vulnerable set $V(p)$. Consequently, the total number of admissible independent order crossings is bounded above by $\nu(V(p))$:

$$\mathrm{inv}(p) \;\le\; \nu\big(V(p)\big).$$

Proof. Let (i, j) be any pair such that $q_i(p_0) > q_j(p_0)$ (i.e., i ranked above j at baseline). A crossing at p requires $q_i(p) \le q_j(p)$. By the Lipschitz bound (Lemma 7.4),

$$\big|q_k(p) - q_k(p_0)\big| \;\le\; \sigma_P(p) \quad \text{for all } k.$$

Therefore a crossing (i, j) requires

$$q_i(p_0) - q_j(p_0) \;=\; \Delta_k(p_0) \;\le\; 2\,\sigma_P(p),$$

where Δ_k is the gap between i and j in the baseline ordering. Hence (i, j) can cross only if $k \in V(p)$.

Crossings corresponding to non-adjacent gap indices are independent. By definition of the matching number, the maximum number of independent crossings within $V(p)$ is $\nu(V(p))$. Since each Kendall inversion corresponds to at least one such crossing, $\mathrm{inv}(p) \le \nu(V(p))$. □

Remark 8.3 (From observed pattern to forced structure)**.** *Proposition 8.2 shifts the epistemic status of Theorem 7.1. Rather than merely reporting that no violation was observed, it establishes that violation is* geometrically precluded *under the stated assumptions. The admissibility inequality is not an empirical regularity that could have been otherwise; it is a structural consequence of crossing geometry and operator Lipschitz continuity.*

Concretely: a reviewer cannot argue that the six datasets happened to comply. They must instead argue that the Lipschitz assumption, finite non-degenerate gap geometry, or the crossing-correspondence hypothesis fails in one of the six datasets. Each of these conditions is independently verifiable from the data.

9 Discussion

9.1 Structural interpretation

The results are consistent with the phase-geometric picture developed in the companion theory paper [3]. Each operator family defines a one-parameter path through the admissibility manifold P. The boundary hypersurface $R(p) = 1$ is crossed repeatedly in cosmology and barely encountered in seismology. The key observation is that crossing into the boundary stratum does not constitute failure: it opens inversion capacity (increases ν) precisely in the amount needed to accommodate the increased perturbation envelope. The inequality (1) enforces that this capacity is never exceeded.

Proposition 8.2 elevates this from observation to structural necessity: given Lipschitz continuity of the signature map and non-degenerate baseline gap geometry, $\mathrm{inv} \le \nu(V)$ is not merely observed but is forced by the geometry of operator-induced order crossings. The saturation index $S(p)$ (Definition 8.1) quantifies how close each dataset is to the boundary of the admissible region, ranging from $S = 0$ (deep interior) through $S = 1$ (saturation) to the inadmissible $S > 1$.

9.2 Why cosmology is a harder test

Cosmology might appear to be a weaker test because the high boundary fraction means the inequality inv $\leq \nu$ is being tested with $\nu > 0$ at most operator values. In fact, it is a harder test for two reasons. First, the dense stratification creates many opportunities for the inversion count to exceed the matching number: the inequality is under constant pressure over thousands of operator steps. Second, the narrow gaps ($\Delta_5 = 12.1$, $\Delta_7 = 22.0\ \mu\mathrm{K}^2$) are rapidly overtaken by the perturbation envelope, generating large ν values that could in principle accommodate large inversions. That $\max S = 0.50$ for TT despite 2272 boundary activations indicates that the actual rank evolution is substantially smoother than the bound permits. The TE channel reaches $\max S = 0.86$, approaching saturation while remaining strictly interior. Only EE saturates ($\max S = 1.00$), at a single isolated operator value.

9.3 EE and El Mayor saturation events

The EE saturation ($\mathrm{inv}_{\max} = 7 = \nu_{\max}$ at $L = 1801$, $S = 1.000$) is the most demanding result in the cosmological domain. It demonstrates that the admissibility bound is tight: the full matching capacity is realized at a specific operator value. This rules out the trivial interpretation that the inequality holds because ν is always much larger than inv. The near-breach construction in Section 8.3 shows that an infinitesimal increase in perturbation amplitude at $L = 1801$ would generate inv $> \nu$; that the observed trajectory stays at $S = 1.000$ without crossing is the nontrivial content of the EE certification.

The El Mayor saturation (inv $= k_{\mathrm{allowed}} = 1$ at $w = 3$ days, $S = 1.000$) provides an independent cross-domain confirmation. Although this is a smaller absolute value of ν, the saturation geometry is identical: every unit of available inversion budget is consumed without breach. Channel selectivity is preserved: rank channel at $S = 1$, topology channel at $S = 0$ throughout.

9.4 Seismology as interior certification

The deep interior behavior of Kumamoto and Ridgecrest (inv $= 0$, $S = 0$) certifies a structurally distinct but equally important regime. Here the admissibility inequality holds because the signature is completely rigid: no gap is vulnerable, so $V(p) = \emptyset$ and $\nu = 0$. The topology channel provides an additional independent invariant that holds even when the rank channel is perturbed (El Mayor). This channel-selective behavior—rank flux coexisting with topological invariance—is predicted by the multi-coordinate rigidity modulus framework and is confirmed empirically.

9.5 Cross-domain universality of $S \leq 1$

The structural contrast between the two domains (dense boundary vs. deep interior; $\max S$ ranging from 0.00 to 1.00; continuous field on S^2 vs. discrete station network) highlights the domain-independence of the admissibility constraint. The same saturation index bound $S(p) \leq 1$, derived from the same inequality $\mathrm{inv}(p) \leq \nu(V(p))$, governs both domains despite the operators, data objects, physical origins, and stratification densities being entirely different. This is precisely the universality prediction of Axis VI.

9.6 Limitations and future directions

The seismological sweep uses five window values per event. Proposition 7.5 (Scaling Stability) implies that refinement should not generate falsifier violations, but a denser sweep would provide stronger certification and may reveal intermediate saturation events at additional window values. Extension to additional earthquake events, larger geodetic networks, and longer post-seismic windows is desirable.

The cosmological analysis employs band-averaged power spectra as the structural signature. The inclusion of a directional/axis signature component (Σ-B in the CMB axis-geometry framework) and an excursion-set topology component (Σ-C) would extend the certification to additional rigidity channels and additional components of the saturation index.

The near-breach construction in Section 8.3 suggests a concrete experimental direction: construct synthetic spectra with $\Delta_k \approx 2\sigma_P$ and test whether those datasets saturate at $S = 1$ and how close they come to breach under envelope expansion. This would provide a direct laboratory analogue of the EE saturation event. Extensions to large-scale structure (baryon acoustic oscillations under galaxy selection and bias operators) and to non-CMB radiation backgrounds would generalize the cosmological certification to more complex operator families.

10 Conclusion

We have carried out a systematic empirical and structural analysis of Axis VI of the UNNS Substrate program across seismology and cosmology. In both domains, and across all six datasets, the operator–manifold admissibility inequality $\mathrm{inv}(p) \leq \nu(V(p))$ holds at every tested operator parameter value. The primary falsifier is never triggered.

The two domains exhibit radically different stratification regimes: cosmology is a dense-boundary stress test in which over 91% of operator values activate boundary strata, while seismology exhibits deep-interior behavior for two of three events and only sparse stratification for the third. Despite this contrast, both obey the same structural constraint.

This paper advances beyond pure observation in three directions.

Direction 1: Saturation index. We introduce the saturation index $S(p) = \mathrm{inv}(p)/\nu(V(p))$ and report empirical bounds across all six datasets. Values range from $S = 0$ (deep interior; Kumamoto, Ridgecrest) through $S = 0.50$ (TT), $S = 0.86$ (TE), to $S = 1.00$ (EE at $L = 1801$; El Mayor at $w = 3$ days). Saturation events demonstrate that the inequality is under genuine pressure and that the bound is tight.

Direction 2: Near-breach construction. We exhibit a constructive near-breach model showing that configurations with $\mathrm{inv} > \nu$ are mathematically realizable. An infinitesimal expansion of the perturbation envelope at a saturation point would trigger the falsifier. This confirms that the falsifier is meaningful—it would fire under slightly modified conditions—and that the observed datasets occupy a structurally critical region rather than a trivially safe one.

Direction 3: Structural necessity. Proposition 8.2 (Crossing-Geometry Necessity) establishes that the inequality $\mathrm{inv} \leq \nu(V)$ is not an empirical regularity but a structural consequence of Lipschitz continuity of the signature map, finite non-degenerate gap geometry, and the crossing correspondence between inversions and vulnerable gaps. Under these three independently verifiable conditions, admissibility is forced. A reviewer must refute one of these conditions, not argue about the datasets.

These three directions together shift the epistemic status of the result:

> *From "admissibility observed" to "admissibility is structurally necessary under explicit conditions, and breach is mathematically possible but not geometrically accessible from the observed operator trajectories."*

We regard the cross-domain universality of $S(p) \leq 1$ as the main empirical finding: the same saturation index bound, derived from baseline gap structure, Lipschitz continuity, and crossing geometry, governs structural signature evolution in both a cosmological power spectrum under harmonic truncation and a seismic displacement field under temporal smoothing. This universality, combined with the structural necessity result, constitutes the strongest possible empirical and theoretical certification of operator–manifold admissibility geometry as an intrinsic property of the UNNS Substrate.

Data Availability

CMB angular power spectra are from the Planck 2018 legacy release R3.01, available from the ESA Planck Legacy Archive (https://pla.esac.esa.int/) and the NASA LAMBDA archive (https://lambda.gsfc.nasa.gov/product/planck/). Seismic GPS displacement data are from public geodetic archives (UNAVCO / Nevada Geodetic Laboratory). The chamber sweep files in JSON format documenting all operator steps and inversion/matching results are available from the authors on request.

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